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Model-based estimation of numbers-at-length and(or) -age for commercial landings in the Northeastern United States

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Introduction

The landings by age for assessment of species in the Northeast United States are estimated using the "age-length key" approach. The numbers-at-length and -age for sampled otoliths or scales are used with numbers-at-length from a larger set of sampled length measurements in the numbers-at-age estimator. The random sampling of length measurements and otoliths is complicated by logistical constraints of covering any port between Virginia and Maine where commercial landings are made. Landings at a subset of ports is a dominant portion of the total landings. Nevertheless, the general sampling protocol consists of determining a port to visit, market categories for which length and otoliths are desired and trip-specific landings at the port available for sampling.

The important attributes (e.g., unbiasedness and variance) of the numbers-at-age estimator depend on the randomization methods used to determine which fish enter the length and otolith samples when design-based inferences is desired (Särndal et al. 1992). Model-based inferences depend on the how closely model assumptions match the reality of the data-generating mechanism (Valliant et al. 2000). Estimators for numbers-at-age using the age-length key approach have been derived for a two-phase design where there is a simple random sample without replacement (SRS) for a length sample at the first phase and a SRS within each length class for otoliths at the second phase (Southward 1976; Kimura 1977).

The estimators I present below for numbers-at-length and(or) -age are derived with respect to model-based inference, but are also appropriate for design-based inference using a specific type of stratified multi-phase design. I explicitly assume a model within each market category of each trip where the length and age category of each fish is random and independent of that for other fish in the trip and market category. I also assume conditional independence of numbers-at-length and(or) -age in trips and in ports. I ignore uncertainty in the estimate of the total number of fish caught in the trip because of inability to estimate its variance and the likelihood that this will be a negligible contributor to the total variance of the numbers-at-age.

Another complication is that there is a higher likelihood of sampling trips with larger landings. When this is ignored, biased prediction of numbers-at-age for the unsampled trips will occur. To correct the estimator, I use a ratio approach where the numbers-at-age are multiplied by the ratio of true total landings to the model-based estimate of the total. In effect, a model-based estimator of numbers-at-age per unit weight is multiplied by the true total weight and I implicitly make a further model assumption that the numbers-at-age per unit weight in the larger landings is the same as that in the unsampled smaller landings.

Model-based estimation of landings-at-length

Suppose that the length category of each fish is random and independent of the random length category of other fish landed in trip j at port i in market category k and that the $(L\times 1)$ indicator vector of the length class for the kth sampled fish, $\mathbf{n}_{h,i,j,k} = (n_{h,i,j,k,1}, \ldots, n_{h,i,j,k,L})^T$, has a multinomial distribution with mean $E(\mathbf{n}_{h,i,j,k}) = \mathbf{p}_{h,i,j}$ and variance $V(\mathbf{n}_{h,i,j,k}) =$

$\text{diag}\{\mathbf{p}_{h,i,j}\} - \mathbf{p}_{h,i,j}\mathbf{p}_{h,i,j}^T$. The vector of numbers-at-length in the sample is

$$\mathbf{n}_{h,i,j} = \sum_{k=1}^{n_{h,i,j}} \mathbf{n}_{h,i,j,k}$$

and a model-based estimator of the numbers-at-length in the trip and market category is

$$\widehat{\mathbf{N}}_{h,i,j} = \frac{N_{h,i,j}}{n_{h,i,j}}\mathbf{n}_{h,i,j} = N_{h,i,j}\widehat{\mathbf{p}}_{h,i,j} \tag{1}$$

where $N_{h,i,j}$ is the total number of fish from the hth market category at the ith port in the jth trip and $\widehat{\mathbf{p}}_{h,i,j}$ is the vector of proportions-at-length in the sample (also an estimator of the proportions-at-length in the trip/market category).

An unbiased prediction-error variance estimator for the numbers-at-length estimator (eq. 1) is

$$\begin{aligned}
\widehat{V}\left(\widehat{\mathbf{N}}_{h,i,j}\right) &= N_{h,i,j}\left(\frac{N_{h,i,j}}{n_{h,i,j}} - 1\right)\frac{\sum_{k=1}^{n_{h,i,j}}\left(\mathbf{n}_{h,i,j,k} - \widehat{\mathbf{p}}_{h,i,j}\right)\left(\mathbf{n}_{h,i,j,k} - \widehat{\mathbf{p}}_{h,i,j}\right)^T}{n_{h,i,j} - 1} \\
&= N_{h,i,j}\left(\frac{N_{h,i,j}}{n_{h,i,j}} - 1\right)\frac{n_{h,i,j}\left[\text{diag}\{\widehat{\mathbf{p}}_{h,i,j}\} - \widehat{\mathbf{p}}_{h,i,j}\widehat{\mathbf{p}}_{h,i,j}^T\right]}{n_{h,i,j} - 1} \qquad (2)
\end{aligned}$$

When boxes of fish are independent and $b_{h,i,j}$ boxes are sampled from $B_{h,i,j}$ total boxes of fish of the hth market category in the trip, an unbiased estimator of the total numbers-at-length is

$$\widehat{\mathbf{N}}_{h,i,j} = \widehat{N}_{h,i,j}\widehat{\mathbf{p}}_{h,i,j} \tag{3}$$

where

$$\widehat{N}_{h,i,j} = \frac{B_{h,i,j}}{b_{h,i,j}}\sum_{g=1}^{b_{h,i,j}} N_{h,i,j,g}$$

where $N_{h,i,j,g}$ is the number of fish in the gth box. An unbiased variance estimator is

$$\begin{aligned}
\widehat{V}\left(\widehat{\mathbf{N}}_{h,i,j}\right) &= \widehat{N}_{h,i,j}\left(\frac{\widehat{N}_{h,i,j}}{n_{h,i,j}} + 1\right)\frac{n_{h,i,j}\left[\text{diag}\{\widehat{\mathbf{p}}_{h,i,j}\} - \widehat{\mathbf{p}}_{h,i,j}\widehat{\mathbf{p}}_{h,i,j}^T\right]}{n_{h,i,j} - 1} \\
&\quad + \widehat{V}\left(\widehat{N}_{h,i,j}\right)\left[\widehat{\mathbf{p}}_{h,i,j}\widehat{\mathbf{p}}_{h,i,j}^T - \frac{\text{diag}\{\widehat{\mathbf{p}}_{h,i,j}\} - \widehat{\mathbf{p}}_{h,i,j}\widehat{\mathbf{p}}_{h,i,j}^T}{n_{h,i,j} - 1}\right] \qquad (4)
\end{aligned}$$

where

$$\widehat{V}\left(\widehat{N}_{h,i,j}\right) = B_{h,i,j}\left(\frac{B_{h,i,j}}{b_{h,i,j}} - 1\right)\frac{\sum_{g=1}^{b_{h,i,j}}\left(N_{h,i,j,g} - \overline{N}_{h,i,j}\right)^2}{b_{h,i,j} - 1}$$

is the estimator of the variance of the total numbers of fish in the market category and $\overline{N}_{h,i,j} = \sum_{g=1}^{b_{h,i,j}} N_{h,i,j,g}/b_{h,i,j}$. Notice that at least 2 boxes must be sampled from the market category for unbiased variance estimation ($b_{h,i,j} > 1$).

When we are interested in numbers at length at the quarterly level, the variance of the total numbers per trip may be negligible. Assuming that the numbers at length of the hth market category in each trip is independently distributed in a given quarter, the unbiased estimator of the total numbers at length in the hth market category for the quarter is

$$\widehat{\mathbf{N}}_{h,i} = \frac{T_{h,i}}{t_{h,i}} \sum_{j=1}^{t_{h,i}} \widehat{\mathbf{N}}_{h,i,j} \tag{5}$$

and the variance estimator is

$$\widehat{V}\left(\widehat{\mathbf{N}}_{h,i}\right) = T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}} - 1\right) \frac{\sum_{j=1}^{t_{h,i}} \left(\widehat{\mathbf{N}}_{h,i,j} - \widehat{\overline{\mathbf{N}}}_{h,i}\right)\left(\widehat{\mathbf{N}}_{h,i,j} - \widehat{\overline{\mathbf{N}}}_{h,i}\right)^T}{t_{h,i} - 1} + \frac{T_{h,i}}{t_{h,i}} \sum_{j=1}^{t_{h,i}} \widehat{V}\left(\widehat{\mathbf{N}}_{h,i,j}\right) \tag{6}$$

where $T_{h,i}$ and $t_{h,i}$ are the total and sampled numbers of trips for the hth market category at the ith port and $\widehat{V}\left(\widehat{\mathbf{N}}_{h,i,j}\right)$ is provided by eq. 4.

Assuming independence of ports, the estimator of the numbers-at-length vector over all ports is

$$\widehat{\mathbf{N}}_h = \frac{P_h}{p_h} \sum_{i=1}^{p_h} \widehat{\mathbf{N}}_{h,i}$$

and the variance estimator is

$$\widehat{V}\left(\widehat{\mathbf{N}}_h\right) = P_h\left(\frac{P_h}{p_h} - 1\right) \frac{\sum_{i=1}^{p_h} \left(\widehat{\mathbf{N}}_{h,i} - \widehat{\overline{\mathbf{N}}}_h\right)\left(\widehat{\mathbf{N}}_{h,i} - \widehat{\overline{\mathbf{N}}}_h\right)^T}{p_h - 1} + \frac{P_h}{p_h} \sum_{i=1}^{p_h} \widehat{V}\left(\widehat{\mathbf{N}}_{h,i}\right)$$

The estimator of the numbers-at-length vector over all market categories is just the sum of the category-specific estimators

$$\widehat{\mathbf{N}} = \sum_{h=1}^{M} \widehat{\mathbf{N}}_h$$

and because sampling is independent for each market category, the variance estimator is

$$\widehat{V}\left(\widehat{\mathbf{N}}\right) = \sum_{h=1}^{M} \widehat{V}\left(\widehat{\mathbf{N}}_h\right) \tag{7}$$

The basic estimator I propose to correct for sampling trips larger in landed weight is

$$\widehat{\mathbf{N}}_W = \frac{W}{\widehat{W}} \widehat{\mathbf{N}} \tag{8}$$

where the estimator of the total weight would be

$$\widehat{W} = \sum_{h=1}^{M} \frac{P_h}{p_h} \sum_{i=1}^{p_h} \frac{T_{h,i}}{t_{h,i}} \sum_{j=1}^{t_{h,i}} W_{h,i,j},$$

with variance estimator

$$\widehat{V}\left(\widehat{W}\right) = \sum_{h=1}^{M} P_h \left(\frac{P_h}{p_h} - 1\right) \frac{\sum_{i=1}^{p_h}\left(\widehat{W}_{h,i} - \widehat{\overline{W}}_h\right)^2}{p_h - 1} + \frac{P_h}{p_h}\sum_{i=1}^{p_h} \widehat{V}\left(\widehat{W}_{h,i}\right)$$

where

$$\widehat{V}\left(\widehat{W}_{h,i}\right) = T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}} - 1\right) \frac{\sum_{j=1}^{t_{h,i}}\left(W_{h,i,j} - \widehat{\overline{W}}_{h,i,j}\right)^2}{t_{h,i} - 1}.$$

The approximately unbiased variance estimator for the basic estimator is

$$\widehat{V}\left(\widehat{\mathbf{N}}_W\right) = \left\{\widehat{N}_{W,i}\widehat{N}_{W,j}\left[\frac{\widehat{V}\left(\widehat{W}\right)}{\widehat{W}^2} + \frac{\widehat{Cov}\left(\widehat{N}_i, \widehat{N}_j\right)}{\widehat{N}_i\widehat{N}_j} - \frac{\widehat{Cov}\left(\widehat{W}, \widehat{N}_j\right)}{\widehat{W}\widehat{N}_j} - \frac{\widehat{Cov}\left(\widehat{N}_i, W\right)}{\widehat{N}_i\widehat{W}}\right]\right\}$$

where i and j indicate the (i, j)th element of the matrix and $\widehat{N}_{W,i}$ is the ith element of the basic estimator (eq. 8).

Estimating landings-at-age and -length

Supposing that there are A age classes, the estimator for the numbers-at-age and -length has the same form as the numbers-at-length estimator, we just expand the categories so that we have L categories for every age class and therefore the numbers-at-age and -length vector is $AL \times 1$. Likewise, if we are considering just the numbers-at-age from the age sample we use the same estimators as presented above for the numbers-at-length except we use the age sample size rather than the age sample size.

Estimating the numbers-at-age: the age-length key approach

The usual age-length key estimator for numbers-at-age is the sum of the products of the ratios of the numbers-at-length estimators based on the length sample and age sample, respectively and the numbers at length and -age over all length classes,

$$\widehat{N}_a = \sum_{l=1}^{L} \frac{\widehat{N}_{a,l}}{\widehat{N}_{2,l}}\widehat{N}_{1,l}$$

where $\widehat{N}_{1,l}$ and $\widehat{N}_{2,l}$ are the corresponding number-at-length estimators for the lth length class. The estimator that takes the form of the basic estimator (eq. 8) and corrects for larger landings in sampled trips is

$$\widehat{N}_a = \frac{W}{\widehat{W}}\sum_{l=1}^{L} \frac{\widehat{N}_{a,l}}{\widehat{N}_{2,l}}\widehat{N}_{1,l} \tag{9}$$

and the estimator of the total numbers-at-length from the age sample is

$$\widehat{\mathbf{N}}_2 = \sum_{h=1}^{M} \frac{P_h}{p_h}\sum_{i=1}^{p_h} \frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}} \frac{N_{h,i,j}}{n_{h,i,j,A}}\mathbf{n}_{h,i,j,A}$$

where $\mathbf{n}_{h,i,j,A}$ is the vector of numbers-at-length in the age sample and the variance estimator is

$$\widehat{V}\left(\widehat{\mathbf{N}}_2\right) = \sum_{h=1}^{M} P_h \left(\frac{P_h}{p_h} - 1\right) \frac{\sum_{i=1}^{p_h} \left(\widehat{\mathbf{N}}_{2,h,i} - \widehat{\overline{\mathbf{N}}}_{2,h}\right)\left(\widehat{\mathbf{N}}_{2,h,i} - \widehat{\overline{\mathbf{N}}}_{2,h}\right)^T}{p_h - 1} + \frac{P_h}{p_h} \sum_{i=1}^{p_h} \widehat{V}\left(\widehat{\mathbf{N}}_{2,h,i}\right)$$

where

$$\begin{aligned}\widehat{V}\left(\widehat{\mathbf{N}}_{2,h,i}\right) &= T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}} - 1\right) \frac{\sum_{j=1}^{t_{h,i}} \left(\widehat{\mathbf{N}}_{2,h,i,j} - \widehat{\overline{\mathbf{N}}}_{2,h,i}\right)\left(\widehat{\mathbf{N}}_{2,h,i,j} - \widehat{\overline{\mathbf{N}}}_{2,h,i}\right)^T}{t_{h,i} - 1} \\ &\quad + \frac{T_{h,i}}{t_{h,i}} \sum_{j=1}^{t_{h,i}} \widehat{V}\left(\widehat{\mathbf{N}}_{2,h,i,j}\right),\end{aligned}$$

$$\widehat{V}\left(\widehat{\mathbf{N}}_{2,h,i,j}\right) = N_{h,i,j}\left(\frac{N_{h,i,j}}{n_{h,i,j,A}} - 1\right) \frac{n_{h,i,j,A}\left[\mathrm{diag}\{\widehat{\mathbf{p}}_{h,i,j,A}\} - \widehat{\mathbf{p}}_{h,i,j,A}\widehat{\mathbf{p}}_{h,i,j,A}^T\right]}{n_{h,i,j,A} - 1}$$

$\widehat{\mathbf{p}}_{h,i,j,A}$ is the vector of proportions at length in the age sample and $n_{h,i,j,A}$ is the size of the age sample from the trip and market category.

Variance estimator for the numbers-at-age using the age-length key

The variance-covariance matrix for the estimator of the numbers-at-age will be comprised of variances of the A scalar estimators of total numbers at a given age, $V\left(\widehat{N}_a\right) = Cov(\widehat{N}_a, \widehat{N}_a)$, along the diagonal and off the diagonal will be covariances of each unique pair of estimators, $Cov\left(\widehat{N}_a, \widehat{N}_{a'}\right)$. The variance estimator below will apply to the case where the aging data and length data are all from the same quarter, or for both types of data pooled over multiple quarters if we make the assumption that trips sampled within all the respective quarters are representative of unsampled trips during the same time frame. The

general structure for the variances and covariances via the delta method are

$$
\begin{aligned}
Cov(\widehat{N}_a, \widehat{N}_b) &= \sum_{l=1}^{L}\sum_{m=1}^{L} Cov\left(\frac{W\widehat{N}_{1,l}\widehat{N}_{la}}{\widehat{W}\widehat{N}_{2,l}}, \frac{W\widehat{N}_{1,m}\widehat{N}_{mb}}{\widehat{W}\widehat{N}_{2,m}}\right) \\
&\approx \sum_{l=1}^{L}\sum_{m=1}^{L} R_{la}R_{mb}\left\{\frac{V\left(\widehat{W}\right)}{E\left(\widehat{W}\right)^2} + \frac{Cov\left(\widehat{N}_{1,l}, \widehat{N}_{1,m}\right)}{E\left(\widehat{N}_{1,l}\right)E\left(\widehat{N}_{1,m}\right)} + \frac{Cov\left(\widehat{N}_{2,l}, \widehat{N}_{2,m}\right)}{E\left(\widehat{N}_{2,l}\right)E\left(\widehat{N}_{2,m}\right)}\right. \\
&\quad + \frac{Cov\left(\widehat{N}_{la}, \widehat{N}_{mb}\right)}{E\left(\widehat{N}_{la}\right)E\left(\widehat{N}_{mb}\right)} + \frac{Cov\left(\widehat{N}_{1,l}, \widehat{N}_{mb}\right)}{E\left(\widehat{N}_{1,l}\right)E\left(\widehat{N}_{mb}\right)} + \frac{Cov\left(\widehat{N}_{la}, \widehat{N}_{1,m}\right)}{E\left(\widehat{N}_{la}\right)E\left(\widehat{N}_{1,m}\right)} \\
&\quad + \frac{Cov\left(\widehat{N}_{2,l}, \widehat{W}\right)}{E\left(\widehat{N}_{2,l}\right)E\left(\widehat{W}\right)} + \frac{Cov\left(\widehat{W}, \widehat{N}_{2,m}\right)}{E\left(\widehat{W}\right)E\left(\widehat{N}_{2,m}\right)} - \frac{Cov\left(\widehat{N}_{2,l}, \widehat{N}_{mb}\right)}{E\left(\widehat{N}_{2,l}\right)E\left(\widehat{N}_{mb}\right)} \\
&\quad - \frac{Cov\left(\widehat{N}_{la}, \widehat{N}_{2,m}\right)}{E\left(\widehat{N}_{la}\right)E\left(\widehat{N}_{2,m}\right)} - \frac{Cov\left(\widehat{N}_{1,l}, \widehat{N}_{2,m}\right)}{E\left(\widehat{N}_{l}\right)E\left(\widehat{N}_{2,m}\right)} - \frac{Cov\left(\widehat{N}_{2,l}, \widehat{N}_{1,m}\right)}{E\left(\widehat{N}_{2,l}\right)E\left(\widehat{N}_{m}\right)} \\
&\quad - \frac{Cov\left(\widehat{N}_{1,l}, \widehat{W}\right)}{E\left(\widehat{N}_{1,l}\right)E\left(\widehat{W}\right)} - \frac{Cov\left(\widehat{W}, \widehat{N}_{1,m}\right)}{E\left(\widehat{W}\right)E\left(\widehat{N}_{1,m}\right)} - \frac{Cov\left(\widehat{N}_{al}, \widehat{W}\right)}{E\left(\widehat{N}_{al}\right)E\left(\widehat{W}\right)} \\
&\quad \left. - \frac{Cov\left(\widehat{W}, \widehat{N}_{mb}\right)}{E\left(\widehat{W}\right)E\left(\widehat{N}_{mb}\right)}\right\} \qquad (10)
\end{aligned}
$$

where

$$R_{la} = \frac{WE\left(\widehat{N}_{1,l}\right)E\left(\widehat{N}_{la}\right)}{E\left(\widehat{W}\right)E\left(\widehat{N}_{2,l}\right)}$$

An asymptotically unbiased variance estimator uses estimators in the above equation. For $\widehat{C}ov\left(\widehat{N}_{la}, \widehat{N}_{mb}\right)$, $\widehat{C}ov\left(\widehat{N}_{1,l}, \widehat{N}_{1,m}\right)$ and $\widehat{C}ov\left(\widehat{N}_{2,l}, \widehat{N}_{2,m}\right)$ the elements are available from the variance estimator(s) (eq. 7) for the vectors of numbers at length and age, numbers at length from the length sample and numbers at length from the age sample. Estimators for other

covariance terms in eq. 10 are

$$\begin{aligned}
\widehat{Cov}\left(\widehat{N}_{1,l},\widehat{N}_{m,b}\right) &= \sum_{h=1}^{M}\left\{P_h\left(\frac{P_h}{p_h}-1\right)\frac{\sum_{i=1}^{p_h}\left(\widehat{N}_{1,h,i,l}-\widehat{\overline{N}}_{1,h,l}\right)\left(\widehat{N}_{h,i,m,b}-\widehat{\overline{N}}_{h,m,b}\right)}{p_h-1}\right.\\
&+\frac{P_h}{p_h}\sum_{i=1}^{p_h}\left[T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{N}_{1,h,i,j,l}-\widehat{\overline{N}}_{1,h,i,l}\right)\left(\widehat{N}_{h,i,j,m,b}-\widehat{\overline{N}}_{h,i,m,b}\right)}{t_{h,i}-1}\right.\\
&+\frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}}\left(N_{h,i,j}+n_{L,h,i,j}-n_{A,h,i,j}\right)\left(\frac{N_{h,i,j}}{n_{L,h,i,j}}-1\right)\\
&\left.\left.\times\frac{\sum_{a=1}^{A}\widehat{Cov}\left(n_{h,i,j,l,a},n_{h,i,j,m,b}\right)}{n_{A,h,i,j}-1}\right]\right\},
\end{aligned}$$

$$\begin{aligned}
\widehat{Cov}\left(\widehat{N}_{2,l},\widehat{N}_{m,b}\right) &= \sum_{h=1}^{M}\left\{P_h\left(\frac{P_h}{p_h}-1\right)\frac{\sum_{i=1}^{p_h}\left(\widehat{N}_{2,h,i,l}-\widehat{\overline{N}}_{2,h,l}\right)\left(\widehat{N}_{h,i,m,b}-\widehat{\overline{N}}_{h,m,b}\right)}{p_h-1}\right.\\
&+\frac{P_h}{p_h}\sum_{i=1}^{p_h}\left[T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{N}_{2,h,i,j,l}-\widehat{\overline{N}}_{2,h,i,l}\right)\left(\widehat{N}_{h,i,j,m,b}-\widehat{\overline{N}}_{h,i,m,b}\right)}{t_{h,i}-1}\right.\\
&\left.\left.+\frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}}N_{h,i,j}\left(\frac{N_{h,i,j}}{n_{A,h,i,j}}-1\right)\frac{\sum_{a=1}^{A}\widehat{Cov}\left(n_{h,i,j,l,a},n_{h,i,j,m,b}\right)}{n_{A,h,i,j}-1}\right]\right\},
\end{aligned}$$

$$\begin{aligned}
\widehat{Cov}\left(\widehat{N}_{1,l},\widehat{N}_{2,m}\right) &= \sum_{h=1}^{M}\left\{P_h\left(\frac{P_h}{p_h}-1\right)\frac{\sum_{i=1}^{p_h}\left(\widehat{N}_{1,h,i,l}-\widehat{\overline{N}}_{1,h,l}\right)\left(\widehat{N}_{2,h,i,m}-\widehat{\overline{N}}_{2,h,m}\right)}{p_h-1}\right.\\
&+\frac{P_h}{p_h}\sum_{i=1}^{p_h}\left[T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{N}_{1,h,i,j,l}-\widehat{\overline{N}}_{1,h,i,l}\right)\left(\widehat{N}_{2,h,i,j,m}-\widehat{\overline{N}}_{2,h,i,m}\right)}{t_{h,i}-1}\right.\\
&\left.\left.+\frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}}\left(N_{h,i,j}+n_{L,h,i,j}-n_{A,h,i,j}\right)\left(\frac{N_{h,i,j}}{n_{L,h,i,j}}-1\right)\frac{\widehat{Cov}\left(n_{h,i,j,l},n_{h,i,j,m}\right)}{n_{A,h,i,j}-1}\right]\right\},
\end{aligned}$$

$$\begin{aligned}
\widehat{Cov}\left(\widehat{W},\widehat{N}_{1,m}\right) &= \sum_{h=1}^{M}\left[P_h\left(\frac{P_h}{p_h}-1\right)\frac{\sum_{i=1}^{p_h}\left(\widehat{W}_{h,i}-\widehat{\overline{W}}_{h}\right)\left(\widehat{N}_{1,h,i,m}-\widehat{\overline{N}}_{1,h,m}\right)}{p_h-1}\right.\\
&\left.+\frac{P_h}{p_h}\sum_{i=1}^{p_h}T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{W}_{h,i,j}-\widehat{\overline{W}}_{h,i}\right)\left(\widehat{N}_{1,h,i,j,m}-\widehat{\overline{N}}_{1,h,i,m}\right)}{t_{h,i}-1}\right]
\end{aligned}$$

$$\widehat{Cov}\left(\widehat{W},\widehat{N}_{2,m}\right) = \sum_{h=1}^{M}\left[P_h\left(\frac{P_h}{p_h}-1\right)\frac{\sum_{i=1}^{p_h}\left(\widehat{W}_{h,i}-\widehat{\overline{W}}_h\right)\left(\widehat{N}_{2,h,i,m}-\widehat{\overline{N}}_{2,h,m}\right)}{p_h-1}\right.$$
$$\left.+\frac{P_h}{p_h}\sum_{i=1}^{p_h}T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{W}_{h,i,j}-\widehat{\overline{W}}_{h,i}\right)\left(\widehat{N}_{2,h,i,j,m}-\widehat{\overline{N}}_{2,h,i,m}\right)}{t_{h,i}-1}\right],$$

and

$$\widehat{Cov}\left(\widehat{W},\widehat{N}_{m,b}\right) = \sum_{h=1}^{M}\left[P_h\left(\frac{P_h}{p_h}-1\right)\frac{\sum_{i=1}^{p_h}\left(\widehat{W}_{h,i}-\widehat{\overline{W}}_h\right)\left(\widehat{N}_{h,i,m,b}-\widehat{\overline{N}}_{h,m,b}\right)}{p_h-1}\right.$$
$$\left.+\frac{P_h}{p_h}\sum_{i=1}^{p_h}T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{W}_{h,i,j}-\widehat{\overline{W}}_{h,i}\right)\left(\widehat{N}_{h,i,j,m,b}-\widehat{\overline{N}}_{h,i,m,b}\right)}{t_{h,i}-1}\right]$$

where

$$\widehat{Cov}\left(n_{h,i,j,l,a},n_{h,i,j,m,b}\right)=\begin{cases}-n_{A,h,i,j}\widehat{p}_{h,i,j,l,a}\widehat{p}_{h,i,j,m,b} & \text{for } l\neq m \text{ and } a\neq b\\ n_{A,h,i,j}\widehat{p}_{h,i,j,l,a}(1-\widehat{p}_{h,i,j,l,a}) & \text{otherwise}\end{cases},$$

$$\widehat{Cov}\left(n_{h,i,j,l},n_{h,i,j,m}\right)=\begin{cases}-n_{A,h,i,j}\widehat{p}_{h,i,j,l}\widehat{p}_{h,i,j,m} & \text{for } l\neq m\\ n_{A,h,i,j}\widehat{p}_{h,i,j,l}(1-\widehat{p}_{h,i,j,l}) & \text{otherwise}\end{cases},$$

$\widehat{p}_{h,i,j,a,l}=n_{h,i,j,a,l}/n_{A,h,i,j}$ and $\widehat{p}_{h,i,j,l}=n_{h,i,j,l}/n_{A,h,i,j}$ are the proportions at length and age and proportions at length, respectively, in the age sample.

Variance estimator when the age-length key is applied to a new length sample

The variances and covariances of the estimators for the numbers-at-age derived from applying an age-length key derived from one sample to a numbers-at-length estimate derived from independent catch data are similar to those previously presented for the case when the age-length key and length sample come from the same catches except that some of the component covariance terms are unbiased rather than approximations or are zero because of independent sampling. The variance estimator for the estimator of number at a given age has the same form as eq. 10.

Alternative model-based estimation

It may occur that the number of trips sampled at port i for the market category h is insufficient for variance estimation (i.e., $t_{h,i}=1$). An alternative assumption is that numbers-at-length or -age for each market category in each trip landed within a region (set of ports) are independently distributed. In such case, h in the previous equations would index trips in the region and $T_{h,i}$ and $t_{h,i}$ would designate the total number of trips and number of trips sampled in the ith region. This alternative model implies stratification by region. Under these assumptions the (non-weight corrected) estimator of the total numbers-at-length from

the length sample ($\widehat{\mathbf{N}}_1$) and age sample ($\widehat{\mathbf{N}}_2$), numbers at length and age over all M market categories and R regions all have the form:

$$\widehat{\mathbf{N}} = \sum_{h=1}^{M}\sum_{i=1}^{R}\widehat{\mathbf{N}}_{h,i}$$

where

$$\widehat{\mathbf{N}}_{h,i} = \frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}}\widehat{\mathbf{N}}_{h,i,j}$$

and the variance is

$$\widehat{V}\left(\widehat{\mathbf{N}}\right) = \sum_{h=1}^{M}\sum_{i=1}^{R}\widehat{V}\left(\widehat{\mathbf{N}}_{h,i}\right)$$

where $\widehat{V}\left(\widehat{\mathbf{N}}_{h,i}\right)$ is provided by eq. 6. The total weight estimator is identical except that the total weight of each market category in each trip is known,

$$\widehat{W}_{h,i} = \frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}}W_{h,i,j}.$$

When estimating the numbers-at-age using an age-length key, the variances and covariances of numbers-at-age have the same form (eq. 10), but

$$\begin{aligned}\widehat{Cov}\left(\widehat{N}_{1,l},\widehat{N}_{m,b}\right) &= \sum_{h=1}^{M}\sum_{i=1}^{R}\left[T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{N}_{1,h,i,j,l}-\widehat{\overline{N}}_{1,h,i,l}\right)\left(\widehat{N}_{h,i,j,m,b}-\widehat{\overline{N}}_{h,i,m,b}\right)}{t_{h,i}-1}\right.\\ &\quad+\frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}}(N_{h,i,j}+n_{L,h,i,j}-n_{A,h,i,j})\left(\frac{N_{h,i,j}}{n_{L,h,i,j}}-1\right)\\ &\quad\left.\times\frac{\sum_{a=1}^{A}\widehat{Cov}\left(n_{h,i,j,l,a},n_{h,i,j,m,b}\right)}{n_{A,h,i,j}-1}\right],\end{aligned}$$

$$\begin{aligned}\widehat{Cov}\left(\widehat{N}_{2,l},\widehat{N}_{m,b}\right) &= \sum_{h=1}^{M}\sum_{i=1}^{R}\left[T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{N}_{2,h,i,j,l}-\widehat{\overline{N}}_{2,h,i,l}\right)\left(\widehat{N}_{h,i,j,m,b}-\widehat{\overline{N}}_{h,i,m,b}\right)}{t_{h,i}-1}\right.\\ &\quad\left.+\frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}}N_{h,i,j}\left(\frac{N_{h,i,j}}{n_{A,h,i,j}}-1\right)\frac{\sum_{a=1}^{A}\widehat{Cov}\left(n_{h,i,j,l,a},n_{h,i,j,m,b}\right)}{n_{A,h,i,j}-1}\right],\end{aligned}$$

$$\begin{aligned}\widehat{Cov}\left(\widehat{N}_{1,l},\widehat{N}_{2,m}\right) &= \sum_{h=1}^{M}\sum_{i=1}^{R}\left[T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(\widehat{N}_{1,h,i,j,l}-\widehat{\overline{N}}_{1,h,i,l}\right)\left(\widehat{N}_{2,h,i,j,m}-\widehat{\overline{N}}_{2,h,i,m}\right)}{t_{h,i}-1}\right.\\ &\quad\left.+\frac{T_{h,i}}{t_{h,i}}\sum_{j=1}^{t_{h,i}}(N_{h,i,j}+n_{L,h,i,j}-n_{A,h,i,j})\left(\frac{N_{h,i,j}}{n_{L,h,i,j}}-1\right)\frac{\widehat{Cov}\left(n_{h,i,j,l},n_{h,i,j,m}\right)}{n_{A,h,i,j}-1}\right],\end{aligned}$$

$$\widehat{Cov}\left(\widehat{W},\widehat{N}_{1,m}\right) = \sum_{h=1}^{M}\sum_{i=1}^{R} T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(W_{h,i,j}-\widehat{\overline{W}}_{h,i}\right)\left(\widehat{N}_{1,h,i,j,m}-\widehat{\overline{N}}_{1,h,i,m}\right)}{t_{h,i}-1},$$

$$\widehat{Cov}\left(\widehat{W},\widehat{N}_{2,m}\right) = \sum_{h=1}^{M}\sum_{i=1}^{R} T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(W_{h,i,j}-\widehat{\overline{W}}_{h,i}\right)\left(\widehat{N}_{h,i,j,2,m}-\widehat{\overline{N}}_{h,i,2,m}\right)}{t_{h,i}-1}.$$

and

$$\widehat{Cov}\left(\widehat{W},\widehat{N}_{m,b}\right) = \sum_{h=1}^{M}\sum_{i=1}^{R} T_{h,i}\left(\frac{T_{h,i}}{t_{h,i}}-1\right)\frac{\sum_{j=1}^{t_{h,i}}\left(W_{h,i,j}-\widehat{\overline{W}}_{h,i}\right)\left(\widehat{N}_{h,i,j,m,b}-\widehat{\overline{N}}_{h,i,m,b}\right)}{t_{h,i}-1}.$$

Atlantic Cod example

For this example I estimated numbers-at-age using both the age-length key estimator as well as the basic numbers-at-age estimator. I applied the estimators to Atlantic cod landings data from the first quarter of 2003 and I used 10cm length bins for the length classes. I used the alternative estimation approach where there is a single region and all trips are randomly sampled within the quarter because there were several ports with only one trip sampled. I excluded from the set of sampled trips all trips where length samples were taken but no age samples were taken.

For this set of data the estimators performed similarly in terms of point estimates and precision estimates (Table 1). The estimated numbers-at-age were slightly higher for the younger ages and slightly lower for the older ages using the basic estimator (eq. 8).

Likewise the correlation of estimated numbers-at-age were similar between estimators with high correlation among adjacent middle age classes (Tables 2 and 3). Most of the correlations were positive, but there was stron negative correlation of age 2 and 3 landings estimates with landings of older fish. There is also change in the sign of correlation among age 4 fish with age 8-10 fish between estimators.

Witch Flounder Example

For this example, I also estimated numbers-at-age for witch flounder using both the age-length key estimator as well as the basic numbers-at-age estimator. I applied the estimators for each quarter of 2003 and I used 1cm length bins for the length classes. I used the alternative estimation approach where there is a single region and all trips are randomly sampled within the quarter because there were several ports with only one trip sampled. See Tables 4 to 10 for results.

Discussion

That such a large subsample of lengths are aged is likely the cause of the similar precision of the two different estimators, but the size of the length bins could affect the relative

precision of the two estimators. The relationship of the precision of these estimators to sample sizes is worth further study and would compliment our understanding of the precision of analogous estimators in two-phase designs (Kimura 1977; Smith 1989).

The application of design-based estimators to data clearly not collected through the design implies an embracing of model-based inference (e.g., Valliant et al. 2000). This can be acceptable if we recognize that the inference is not design-based and the estimator reflects an appropriate model. More specifically, some of the estimators provided here are appropriate for a design where there is a SRS of trips, a SRS of length measurements within a market category in a sampled trip and finally a SRS of otoliths from the sample chosen for length measurements. However, I made assumptions of conditional independence at each sampling phase to derive the estimators so that they would be unbiased with respect to model-based inference if those assumptions are valid. The overall model reflects measurements with multiple sources of error. For example, the estimate of total number at length l for a given market category in a trip can be viewed as a measurement of the true total number at length l (in the market category) with error. Furthermore, we can estimate this measurement error with the data components used to make the measurement (the number of fish sampled in the trip with length l in the market category). Similarly, the estimate of total number at length l at a port may be viewed as a measurement with error where there pieces of data used to make the measurement (the trip-specific total numbers at length l) are also measured with error and the precision of the measurements for each trip can be assumed to be different. As such, we have different measurement error components (trip- and port-levels) for an overall estimate (measurement) of the total numbers caught at length l.

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Table 1. Estimated numbers-at-age of Atlantic cod landed for the first quarter of 2003 using the age-length key (eq. 9) and using the basic estimator (eq. 8).

	eq. 8	CV	eq. 9	CV
1	0.00		0.00	
2	53933.32	0.32	58061.08	0.32
3	268362.24	0.22	285023.68	0.25
4	411939.89	0.10	406452.85	0.12
5	79070.21	0.16	73197.89	0.18
6	45602.00	0.33	39666.58	0.35
7	14010.58	0.26	12017.92	0.32
8	2274.38	0.32	1754.06	0.35
9	3189.88	0.34	2530.83	0.35
10	1722.56	0.42	1413.71	0.43
11	61.16	1.02	47.62	1.03

Table 2. Estimated correlation of numbers-at-age for Atlantic cod landed in the first quarter of 2003 based on the the basic estimator (eq. 8).

	1	2	3	4	5	6	7	8	9	10	11
1											
2		1.00	0.36	−0.39	−0.08	−0.22	−0.15	0.12	0.07	0.09	0.04
3		0.36	1.00	−0.49	−0.54	−0.87	−0.76	−0.23	−0.23	−0.16	−0.05
4		−0.39	−0.49	1.00	0.23	0.45	0.31	−0.08	−0.08	−0.01	0.01
5		−0.08	−0.54	0.23	1.00	0.44	0.59	0.54	0.48	0.33	0.16
6		−0.22	−0.87	0.45	0.44	1.00	0.74	0.14	0.14	0.07	0.03
7		−0.15	−0.76	0.31	0.59	0.74	1.00	0.39	0.27	0.31	0.06
8		0.12	−0.23	−0.08	0.54	0.14	0.39	1.00	0.55	0.51	0.17
9		0.07	−0.23	−0.08	0.48	0.14	0.27	0.55	1.00	0.44	0.05
10		0.09	−0.16	−0.01	0.33	0.07	0.31	0.51	0.44	1.00	0.05
11		0.04	−0.05	0.01	0.16	0.03	0.06	0.17	0.05	0.05	1.00

Table 3. Estimated correlation of numbers-at-age for Atlantic cod landed in the first quarter of 2003 based on the age-length key estimator (eq. 9).

	1	2	3	4	5	6	7	8	9	10	11
1											
2		1.00	0.37	−0.38	−0.12	−0.27	−0.25	0.09	0.06	0.07	0.04
3		0.37	1.00	−0.69	−0.65	−0.88	−0.84	−0.31	−0.27	−0.18	−0.07
4		−0.38	−0.69	1.00	0.45	0.63	0.58	0.07	0.04	0.08	0.04
5		−0.12	−0.65	0.45	1.00	0.55	0.64	0.57	0.48	0.33	0.15
6		−0.27	−0.88	0.63	0.55	1.00	0.81	0.23	0.19	0.10	0.05
7		−0.25	−0.84	0.58	0.64	0.81	1.00	0.36	0.26	0.25	0.05
8		0.09	−0.31	0.07	0.57	0.23	0.36	1.00	0.59	0.45	0.18
9		0.06	−0.27	0.04	0.48	0.19	0.26	0.59	1.00	0.47	0.08
10		0.07	−0.18	0.08	0.33	0.10	0.25	0.45	0.47	1.00	0.06
11		0.04	−0.07	0.04	0.15	0.05	0.05	0.18	0.08	0.06	1.00

Table 4. Estimated numbers-at-age of witch flounder landed for the first quarter of 2003 using the age-length key (eq. 9) and using the basic estimator (eq. 8).

	eq. 8	CV	eq. 9	CV
4	0.00		0.00	
5	14580.11	0.87	10452.11	0.96
6	64000.89	0.70	26853.18	0.91
7	473307.51	0.31	630960.91	0.38
8	389045.88	0.13	365431.88	0.19
9	266913.96	0.18	177688.65	0.21
10	172944.85	0.19	187650.14	0.22
11	74397.03	0.47	69071.19	0.48
12	15574.40	0.46	8445.19	0.60
13	892.06	1.04	668.54	1.16
14	19182.54	0.50	14924.89	0.50
15	712.13	0.72	442.72	0.86
16	446.03	1.04	357.38	1.16
17	0.00		0.00	
18	608.22	1.05	372.86	1.18
19	0.00		0.00	
20	0.00		0.00	
21	0.00		0.00	
22	1297.54	0.75	748.24	0.70
24	0.00		0.00	
25	0.00		0.00	
26	266.10	1.05	202.74	1.05
27	0.00		0.00	
28	0.00		0.00	
29	0.00		0.00	
30	405.48	1.06	304.11	1.16
31	202.74	1.06	202.74	1.06
33	0.00		0.00	

Table 5. Estimated numbers-at-age of witch flounder landed for the second quarter of 2003 using the age-length key (eq. 9) and using the basic estimator (eq. 8).

	eq. 8	CV	eq. 9	CV
4	0.00		0.00	
5	53494.56	0.91	38849.68	1.11
6	461411.40	0.27	503668.50	0.36
7	385006.16	0.08	559307.55	0.07
8	325558.06	0.27	342015.07	0.20
9	363257.71	0.25	350090.96	0.27
10	255970.81	0.22	124646.83	0.24
11	39841.10	0.53	29745.60	0.55
12	88168.83	0.53	41237.47	0.33
13	12400.12	0.66	7584.13	0.58
14	23492.45	0.62	11975.48	0.66
15	209.08	1.07	159.35	1.11
16	549.31	0.56	440.39	0.62
17	491.03	0.93	389.77	0.92
18	4230.47	0.65	4047.39	0.79
19	91.09	1.07	65.25	1.03
20	0.00		0.00	
21	0.00		0.00	
22	164.98	0.77	130.93	0.79
24	0.00		0.00	
25	0.00		0.00	
26	0.00		0.00	
27	36.44	1.07	19.27	1.17
28	0.00		0.00	
29	0.00		0.00	
30	0.00		0.00	
31	0.00		0.00	
33	0.00		0.00	

Table 6. Estimated numbers-at-age of witch flounder landed for the third quarter of 2003 using the age-length key (eq. 9) and using the basic estimator (eq. 8).

	eq. 8	CV	eq. 9	CV
4	85219.69	0.62	11418.23	0.52
5	191620.65	0.10	232504.25	0.15
6	519732.05	0.28	738324.68	0.43
7	424764.89	0.33	537957.39	0.21
8	721518.24	0.16	583349.87	0.09
9	191780.66	0.22	143893.20	0.24
10	235518.86	0.21	151741.08	0.47
11	12829.65	0.60	8369.63	0.65
12	9706.36	0.64	7293.23	0.71
13	36850.27	0.97	27343.87	1.05
14	21754.22	0.61	12611.61	0.56
15	764.37	0.67	555.32	0.61
16	1816.12	0.80	1480.17	0.73
17	159.57	1.09	66.41	1.13
18	4761.88	0.66	2556.88	0.67
19	0.00		0.00	
20	511.01	0.68	306.76	0.73
21	83.81	1.09	39.04	1.28
22	867.70	0.64	479.41	0.62
24	0.00		0.00	
25	0.00		0.00	
26	0.00		0.00	
27	0.00		0.00	
28	0.00		0.00	
29	176.08	1.08	145.06	1.11
30	0.00		0.00	
31	0.00		0.00	
33	0.00		0.00	

Table 7. Estimated numbers-at-age of witch flounder landed for the fourth quarter of 2003 using the age-length key (eq. 9) and using the basic estimator (eq. 8).

	eq. 8	CV	eq. 9	CV
4	49369.26	0.60	36443.11	0.73
5	325886.67	0.10	312059.04	0.12
6	296697.74	0.14	488098.10	0.21
7	344108.49	0.08	411194.52	0.13
8	194086.86	0.23	136846.82	0.41
9	167101.86	0.26	108682.55	0.38
10	119889.26	0.35	47977.54	0.54
11	49654.86	0.66	14074.12	0.52
12	9104.16	0.92	4944.42	0.80
13	3928.22	1.04	1650.20	1.07
14	2808.11	0.42	2029.09	0.40
15	478.87	0.67	338.44	0.66
16	619.11	0.69	393.46	0.72
17	394.72	0.59	223.20	0.58
18	1464.70	0.51	916.70	0.53
19	165.65	0.71	100.06	0.71
20	121.71	0.88	65.63	0.80
21	142.96	1.05	131.29	1.02
22	125.83	0.57	71.89	0.58
24	53.87	1.05	19.35	0.89
25	20.04	1.05	10.36	1.07
26	0.00		0.00	
27	0.00		0.00	
28	20.04	1.05	15.89	1.03
29	0.00		0.00	
30	0.00		0.00	
31	20.04	1.05	11.49	1.15
33	53.87	1.05	19.62	0.95

Table 8. Estimated numbers-at-age of witch flounder landed in 2003 using the age-length key (eq. 9) and using the basic estimator (eq. 8).

	eq. 8	CV	eq. 9	CV
4	134588.95	0.45	47861.35	0.57
5	585581.99	0.11	593865.07	0.11
6	1341842.08	0.15	1756944.46	0.22
7	1627187.06	0.13	2139420.37	0.13
8	1630209.03	0.10	1427643.63	0.09
9	989054.19	0.12	780355.35	0.15
10	784323.78	0.12	512015.60	0.18
11	176722.64	0.30	121260.54	0.32
12	122553.75	0.39	61920.31	0.26
13	54070.67	0.68	37246.74	0.78
14	67237.32	0.33	41541.07	0.31
15	2164.45	0.38	1495.83	0.39
16	3430.57	0.47	2671.41	0.46
17	1045.33	0.52	679.38	0.57
18	11065.27	0.39	7893.83	0.47
19	256.74	0.59	165.31	0.59
20	632.72	0.58	372.39	0.61
21	226.77	0.78	170.33	0.84
22	2456.06	0.46	1430.47	0.43
24	53.87	1.05	19.35	0.89
25	20.04	1.05	10.36	1.07
26	266.10	1.05	202.74	1.05
27	36.44	1.07	19.27	1.17
28	20.04	1.05	15.89	1.03
29	176.08	1.08	145.06	1.11
30	405.48	1.06	304.11	1.16
31	222.79	0.97	214.23	1.00
33	53.87	1.05	19.62	0.95

Table 9. Estimated correlation of numbers-at-age (ages 4-17 with ages 4-33) for witch flounder landed in 2003 based on the age-length key estimator (eq. 9).

	4	5	6	7	8	9	10	11	12	13	14	15	16	17
4	1.00	−0.24	0.15	−0.18	0.19	−0.11	−0.12	−0.09	−0.02	−0.16	−0.05	0.01	−0.03	0.05
5	−0.24	1.00	0.07	0.05	−0.13	−0.47	−0.08	0.12	−0.34	0.10	0.03	−0.03	0.00	0.06
6	0.15	0.07	1.00	−0.34	−0.28	0.12	−0.78	−0.16	0.18	−0.80	−0.51	−0.26	−0.33	−0.21
7	−0.18	0.05	−0.34	1.00	−0.21	−0.16	−0.07	−0.57	−0.20	0.37	−0.16	−0.10	0.01	−0.02
8	0.19	−0.13	−0.28	−0.21	1.00	−0.49	0.24	0.31	−0.30	0.33	0.23	0.10	0.00	0.19
9	−0.11	−0.47	0.12	−0.16	−0.49	1.00	0.08	−0.28	0.69	−0.11	−0.11	0.06	0.00	−0.25
10	−0.12	−0.08	−0.78	−0.07	0.24	0.08	1.00	0.46	−0.01	0.68	0.37	0.18	0.28	0.09
11	−0.09	0.12	−0.16	−0.57	0.31	−0.28	0.46	1.00	−0.09	0.10	0.25	0.02	0.15	0.18
12	−0.02	−0.34	0.18	−0.20	−0.30	0.69	−0.01	−0.09	1.00	0.01	0.04	0.20	0.08	−0.16
13	−0.16	0.10	−0.80	0.37	0.33	−0.11	0.68	0.10	0.01	1.00	0.24	0.17	0.20	0.09
14	−0.05	0.03	−0.51	−0.16	0.23	−0.11	0.37	0.25	0.04	0.24	1.00	0.56	0.34	0.36
15	0.01	−0.03	−0.26	−0.10	0.10	0.06	0.18	0.02	0.20	0.17	0.56	1.00	0.42	0.33
16	−0.03	0.00	−0.33	0.01	0.00	0.00	0.28	0.15	0.08	0.20	0.34	0.42	1.00	0.29
17	0.05	0.06	−0.21	−0.02	0.19	−0.25	0.09	0.18	−0.16	0.09	0.36	0.33	0.29	1.00
18	−0.01	0.10	−0.33	0.01	0.11	−0.25	0.18	0.27	0.03	0.19	0.44	0.25	0.48	0.33
19	0.10	−0.03	−0.12	−0.05	0.22	−0.17	0.05	0.10	−0.08	0.05	0.16	0.08	0.13	0.14
20	−0.04	−0.07	−0.30	0.08	−0.06	0.08	0.26	0.08	0.16	0.20	0.31	0.37	0.33	0.08
21	0.08	−0.10	−0.07	−0.03	0.10	0.02	0.05	0.01	0.09	0.05	0.05	0.09	0.06	0.06
22	−0.03	0.05	−0.19	−0.16	0.05	0.07	0.19	0.09	0.12	0.14	0.38	0.46	0.36	0.04
24	0.11	−0.13	−0.02	−0.06	0.13	0.00	0.00	0.00	0.08	0.02	0.02	0.13	0.06	0.07
25	0.09	−0.10	−0.01	−0.05	0.11	0.00	0.00	0.00	0.05	0.01	0.01	0.05	0.01	0.02
26	0.00	0.04	0.01	−0.17	0.04	0.06	0.03	0.06	0.02	0.00	0.11	0.23	−0.05	0.00
27	0.00	0.10	−0.13	0.01	0.12	−0.20	0.07	0.12	−0.19	0.04	0.13	−0.03	−0.01	0.09
28	0.10	−0.10	−0.01	−0.05	0.12	0.00	0.00	0.00	0.05	0.01	0.01	0.05	0.02	0.03
29	−0.05	−0.04	−0.21	0.06	−0.07	0.06	0.19	0.05	0.10	0.13	0.23	0.39	0.23	0.03
30	0.00	0.04	0.01	−0.14	0.05	0.05	0.03	0.03	0.02	0.00	0.05	−0.10	−0.04	0.00
31	0.01	0.04	0.01	−0.16	0.06	0.06	0.03	0.04	0.01	0.00	0.04	−0.11	−0.04	0.00
33	0.10	−0.12	−0.02	−0.06	0.13	0.00	0.00	0.00	0.07	0.02	0.02	0.10	0.05	0.05

Table 10. Estimated correlation of numbers-at-age (ages 18-33 with ages 4-33) for witch flounder landed in 2003 based on the age-length key estimator (eq. 9).

	18	19	20	21	22	24	25	26	27	28	29	30	31	33
4	−0.01	0.10	−0.04	0.08	−0.03	0.11	0.09	0.00	0.00	0.10	−0.05	0.00	0.01	0.10
5	0.10	−0.03	−0.07	−0.10	0.05	−0.13	−0.10	0.04	0.10	−0.10	−0.04	0.04	0.04	−0.12
6	−0.33	−0.12	−0.30	−0.07	−0.19	−0.02	−0.01	0.01	−0.13	−0.01	−0.21	0.01	0.01	−0.02
7	0.01	−0.05	0.08	−0.03	−0.16	−0.06	−0.05	−0.17	0.01	−0.05	0.06	−0.14	−0.16	−0.06
8	0.11	0.22	−0.06	0.10	0.05	0.13	0.11	0.04	0.12	0.12	−0.07	0.05	0.06	0.13
9	−0.25	−0.17	0.08	0.02	0.07	0.00	0.00	0.06	−0.20	0.00	0.06	0.05	0.06	0.00
10	0.18	0.05	0.26	0.05	0.19	0.00	0.00	0.03	0.07	0.00	0.19	0.03	0.03	0.00
11	0.27	0.10	0.08	0.01	0.09	0.00	0.00	0.06	0.12	0.00	0.05	0.03	0.04	0.00
12	0.03	−0.08	0.16	0.09	0.12	0.08	0.05	0.02	−0.19	0.05	0.10	0.02	0.01	0.07
13	0.19	0.05	0.20	0.05	0.14	0.02	0.01	0.00	0.04	0.01	0.13	0.00	0.00	0.02
14	0.44	0.16	0.31	0.05	0.38	0.02	0.01	0.11	0.13	0.01	0.23	0.05	0.04	0.02
15	0.25	0.08	0.37	0.09	0.46	0.13	0.05	0.23	−0.03	0.05	0.39	−0.10	−0.11	0.10
16	0.48	0.13	0.33	0.06	0.36	0.06	0.01	−0.05	−0.01	0.02	0.23	−0.04	−0.04	0.05
17	0.33	0.14	0.08	0.06	0.04	0.07	0.02	0.00	0.09	0.03	0.03	0.00	0.00	0.05
18	1.00	0.16	0.24	0.07	0.14	0.08	0.00	−0.01	0.11	0.01	0.14	−0.02	−0.02	0.07
19	0.16	1.00	0.14	0.18	0.01	0.11	−0.04	0.00	0.04	0.00	0.00	0.00	0.00	0.06
20	0.24	0.14	1.00	0.06	0.23	0.01	−0.01	0.00	0.00	0.00	0.80	0.00	0.00	−0.01
21	0.07	0.18	0.06	1.00	0.04	0.15	0.07	0.00	0.00	0.08	−0.02	0.00	0.00	0.13
22	0.14	0.01	0.23	0.04	1.00	0.05	0.00	−0.12	0.00	0.00	0.07	−0.14	0.53	0.05
24	0.08	0.11	0.01	0.15	0.05	1.00	0.01	0.00	0.00	0.02	0.00	0.00	−0.01	0.96
25	0.00	−0.04	−0.01	0.07	0.00	0.01	1.00	0.00	0.00	0.99	0.00	0.00	0.06	0.02
26	−0.01	0.00	0.00	0.00	−0.12	0.00	0.00	1.00	0.00	0.00	0.00	−0.11	−0.12	0.00
27	0.11	0.04	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
28	0.01	0.00	0.00	0.08	0.00	0.02	0.99	0.00	0.00	1.00	0.00	0.00	0.06	0.02
29	0.14	0.00	0.80	−0.02	0.07	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00
30	−0.02	0.00	0.00	0.00	−0.14	0.00	0.00	−0.11	0.00	0.00	0.00	1.00	−0.10	0.00
31	−0.02	0.00	0.00	0.00	0.53	−0.01	0.06	−0.12	0.00	0.06	0.00	−0.10	1.00	−0.01
33	0.07	0.06	−0.01	0.13	0.05	0.96	0.02	0.00	0.00	0.02	0.00	0.00	−0.01	1.00